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Steady state multiplicity in boiling fluid pipe flow

I. NAOT,† D. R. LEWIN†§ and S. J. WAJC‡
† Department of Chemical Engineering, Technion, Haifa 32000, Israel
‡ IMI—Institute for Research and Development, PO Box 10140, Haifa Bay, Israel

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1. INTRODUCTION

THE PROBLEM considered is that of a pipe in which flows a fluid that may evaporate. A uniform heat flux is supplied along the pipe and is independent of the flow rate inside the pipe. This may be the case in nuclear reactor cores, in electrically heated tubes or as an approximation in cases where the pipe is heated by radiation by high temperature flames. The fluid enters the pipe as a sub-cooled liquid and exits the pipe as a liquid–vapor mixture. An increase in the mass flow rate causes both an increase in the frictional pressure drop and a decrease in the length of the two-phase zone, and in the exit vapor fraction, causing a smaller change in momentum flux. These conflicting effects result in the possibility of a local maximum in the variation of pressure drop with flow rate, as will be shown later.

2. ANALYTICAL MODEL

In order to understand the physics of the phenomenon, we shall first develop a simplified approximation, for which an analytical solution can be derived. This will then be compared with a numerical simulation of the system described. The assumptions of the model are:

(a) Liquid and vapor properties are constants (ΔH_{LV}, ΔV_{LV}, V_L, C_{PL}).
(b) The pressure drop is small and therefore the saturation temperature, T_s, is constant.
(c) The liquid viscosity is constant and the two-phase mixture viscosity is equal to the liquid viscosity.
(d) The flow is turbulent and the friction factor is described by the Blasius approximation: $f_f = 0.079\ Re^{-0.25}$.
(e) The two-phase flow is described by the homogeneous flow model.
(f) The heat input to the pipe, q_{in}, is uniform.
(g) The flow is one dimensional (no radial changes).
(h) Axial heat conduction is neglected.
(i) In the two-phase zone, the fluid is in equilibrium at all points.
(j) Steady state conditions are assumed throughout.

We note that assumptions (a)–(c) above do not apply to the numerical simulations brought in Section 3. The two model equations are:

Energy balance:

$$\dot{m}\frac{dH}{dl} = q_{in} \quad (1)$$

In the liquid zone:

$$\frac{dH}{dl} = C_{PL}\frac{dT}{dl} \quad (1a)$$

In the two-phase zone:

$$\frac{dH}{dl} = \Delta H_{LV}\frac{d\langle x\rangle}{dl} \quad (1b)$$

Momentum balance:

$$\frac{dP}{dl} = \frac{dP_f}{dl} + \frac{dP_a}{dl}. \quad (2)$$

Here, the terms on the right hand side are the frictional pressure gradient:

$$\frac{dP_f}{dl} = -2f_f\frac{\rho u^2}{D}, \quad (3)$$

and the pressure gradient due to acceleration:

$$\frac{dP_a}{dl} = -\frac{d}{dl}(\langle \rho u^2\rangle). \quad (4)$$

These can be expressed in terms of $\dot{m}$: $\rho u = \dot{G} = \dot{m}/A$; $\rho u^2 = \dot{G}^2/\rho = \dot{m}^2/A^2\cdot\langle V\rangle$; $Re = 4\dot{m}/\pi D\mu$. In the two-phase region:

$$\langle V\rangle = V_L + \langle x\rangle\Delta V_{LV},$$

and

$$\frac{d\langle V\rangle}{dl} = \Delta V_{LV}\frac{d\langle x\rangle}{dl}.$$

From (1) and (1b):

$$\frac{d\langle x\rangle}{dl} = \frac{q_{in}}{\dot{m}\Delta H_{LV}}.$$

Substituting into (4) gives:

§ To whom correspondence should be addressed.

NOMENCLATURE

- A pipe cross-sectional area [m^2]
- C a constant, $2 \cdot 0.079\mu^{0.25}/A^2D^{0.75}(4/\pi)^{0.25}$ [$kg^{0.25}\,s^{-0.25}\,m^{-5}$]
- C_{PL} specific heat of liquid [$kJ\,kg^{-1}\,{}^{\circ}C^{-1}$]
- $\tilde{C}_P$ dimensionless subcooling, $C_{PL}(T_s - T_I)/\Delta H_{LV}$
- D pipe diameter [m]
- f_f Blasius friction factor, $0.079\,Re^{-0.25}$
- l pipe length [m]
- l_S position along pipe at which boiling starts [m]
- l_0 total pipe length [m]
- $\dot{m}$ mass flow rate [$kg\,s^{-1}$]
- $\dot{m}^*$ critical mass flow rate, $(q_{in}l_0)/(C_{PL}[T_s - T_I])$
- $\tilde{m}$ dimensionless mass flow rate, $\dot{m}/\dot{m}^*$
- M_L single phase liquid momentum flux, $(\dot{m}^2/A^2)V_L$
- $\dot{G}$ mass flux [$kg\,m^{-2}\,s^{-1}$]
- H enthalpy [$kJ\,kg^{-1}$]
- q_{in} heat input to the pipe [$kW\,m^{-1}$]
- Re Reynolds number
- T temperature [°C]
- T_I inlet temperature [°C]
- T_s saturation temperature [°C]
- u fluid velocity [$m\,s^{-1}$]
- V_L liquid specific volume [$m^3\,kg^{-1}$]
- $\langle V \rangle$ average specific volume of the fluid [$m^3\,kg^{-1}$]
- $\tilde{V}$ dimensionless volume change, $\Delta V_{LV}/V_L$
- $\langle x \rangle$ vapor mass fraction
- $\tilde{X}$ dimensionless required heat of vaporization, $(\dot{m}\Delta H_{LV})/(q_{in}l_0)$
- ΔH_{LV} heat of vaporization [$kJ\,kg^{-1}$]
- ΔP total pressure drop [bar]
- ΔP_a pressure drop due to acceleration [bar]
- ΔP_f pressure drop due to friction [bar]
- ΔP_{fL} pressure drop due to friction for single phase liquid flow [bar]
- ΔV_{LV} specific volume change due to vaporization [$m^3\,kg^{-1}$].

Greek symbols

- μ dynamic viscosity of the fluid [bar s]
- ρ fluid density [$kg\,m^{-3}$].

$$\frac{dP_a}{dl} = -\frac{\dot{m}}{A^2}\frac{q_{in}\Delta V_{LV}}{\Delta H_{LV}} \tag{4a}$$

and when substituted into equation (3) gives:

$$\frac{dP_f}{dl} = -C\dot{m}^{1.75}(V_L + \langle x \rangle \Delta V_{LV}) \tag{5}$$

where C is a constant. At a given flow rate $\dot{m}$ of a fluid entering the pipe at a temperature T_I, boiling will start at point $l = l_S$, where:

$$l_S = \frac{\dot{m}C_{PL}(T_s - T_I)}{q_{in}}. \tag{6}$$

This is a direct result of (1a). The vapor fraction, $\langle x \rangle$, can now be expressed explicitly as a function of l:

$$\langle x \rangle = \frac{q_{in}(l - l_S)}{\dot{m}\Delta H_{LV}} = \frac{q_{in}l}{\dot{m}\Delta H_{LV}} - \frac{C_{PL}(T_s - T_I)}{\Delta H_{LV}}. \tag{7}$$

Integrating equations (3) and (4) over the pipe length l_0, we obtain an expression for the overall pressure drop, defined as the inlet pressure minus the outlet pressure (so it is a positive term). This expression can be presented in a general form by defining the following groups:

(a) Dimensionless subcooling: $\tilde{C}_P = C_{PL}(T_s - T_I)/\Delta H_{LV}$.
(b) Dimensionless required heat of vaporization: $\tilde{X} = \dot{m}\Delta H_{LV}/q_{in}l_0$.
(c) Dimensionless volume change: $\tilde{V} = \Delta V_{LV}/V_L$.
(d) Single phase liquid momentum flux: $M_L = \dot{m}^2V_L/A^2$.
(e) Single phase liquid pressure drop $\Delta P_{fL} \equiv -Cl_0V_L\dot{m}^{1.75}$.
(f) Critical flow rate, $\dot{m}^* = q_{in}l_0/C_{PL}(T_s - T_I)$.
(g) Dimensionless mass flow rate, $\tilde{m} = \dot{m}C_{PL}(T_s - T_I)/q_{in}l_0 = \dot{m}/\dot{m}^* = \tilde{X}\tilde{C}_P$.
(h) Characteristic frictional pressure drop, $\Delta P^*_{fL} \equiv \Delta P_{fL}(\dot{m}^*) = Cl_0V_L\dot{m}^{*1.75}$.
(i) Characteristic momentum flux, $M^*_L \equiv M_L(\dot{m}^*) = \dot{m}^{*2}V_L/A^2$.

Physically, $\dot{m}^*$ is the flow rate at which the exciting fluid is saturated liquid exactly at its boiling point. The general form for the overall pressure drop thus obtained is:

$$\widetilde{\Delta P} = \frac{\Delta P}{\Delta P^*_{fL}} = \left(\frac{\tilde{C}_P\tilde{V}}{2}\right)\tilde{m}^{2.75} - \left(\frac{M^*_L}{\Delta P^*_{fL}}\tilde{C}_P\tilde{V}\right)\tilde{m}^2 + \left(1 - \tilde{C}_P\tilde{V}\right)\tilde{m}^{1.75} + \left(\frac{M^*_L}{\Delta P^*_{fL}}\tilde{C}_P\tilde{V}\right)\tilde{m} + \left(\frac{\tilde{C}_P\tilde{V}}{2}\right)\tilde{m}^{0.75}. \tag{8}$$

This model is valid for $\tilde{C}_P > 0$ (meaning that the entering fluid is subcooled) and for $\tilde{m} \leqslant 1$ (meaning that $\dot{m} \leqslant \dot{m}^*$). For $\tilde{m} > 1$, the pressure drop is expressed by:

$$\widetilde{\Delta P} = \tilde{m}^{1.75}. \tag{9}$$

Equation (8) has negative coefficients: that of $\tilde{m}^2$ is always negative and that of $\tilde{m}^{1.75}$ is negative when $\tilde{C}_P\tilde{V} > 1$. The presence of negative coefficients in a polynomial means that there is the possibility of local extrema. In this case, it means that applying the specified pressure drop may result in more than one steady state flow rate. Differentiation of equation (8) with respect to $\tilde{m}$ yields a condition for the existence of extrema. If $d\widetilde{\Delta P}/d\tilde{m} = 0$ has real roots in the range $0 \leqslant \tilde{m} < 1$, then equation (8) will have a region of multiplicity. Since equation (8) has only two parameters, namely $\tilde{C}_P\tilde{V}$ and $M^*_L/\Delta P^*_{fL}$, it is possible to map the region of parameters for which steady state multiplicity occurs. Figure 1 shows such a map; parameter values above the line will give steady state multiplicity.

3. NUMERICAL SIMULATION

The fundamental equations (1) and (2) were approximated by finite difference equations:

$$\dot{m}\Delta H = q_{in} \cdot \Delta l \tag{10}$$

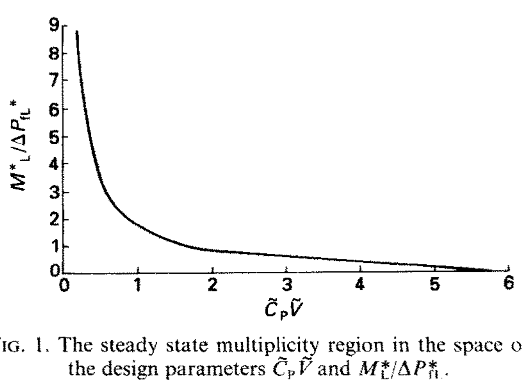

FIG. 1. The steady state multiplicity region in the space of the design parameters $\tilde{C}_P\tilde{V}$ and $M^*_L/\Delta P^*_{fL}$.

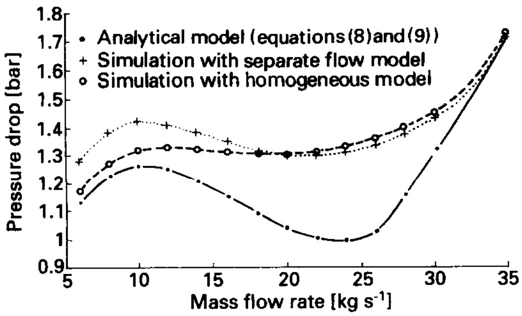

FIG. 2. Variation of total pressure drop with mass flow rate for water–steam flow in the steady state multiplicity region.

$$\Delta P = \left.\frac{\mathrm{d}P_{\mathrm{f}}}{\mathrm{d}l}\right|_{l} \cdot \Delta l + \Delta(\langle \rho u^2 \rangle). \quad (11)$$

The two equations were integrated by summation over pipe segments to yield the pressure profile in a pipe. The fluid properties were calculated at each segment as a function of P and H. For the simulation of water–steam flow, the properties were taken to be those of saturated liquid and vapor at the given pressure in the two-phase zone and at the given temperature in the liquid zone. The properties were interpolated from published steam tables [3]. Thus all the assumptions related to physical properties used in the analytical model (assumptions (a)–(c)) do not apply here. Simulations were carried out using two different models for the local two-phase pressure gradient. In one case, the homogeneous flow model [1, 2] with arithmetically-averaged viscosity was used, while in the other case, the separate flow model [1, 2, 4, 6] was used, with an approximation for the local void fraction [1, 5].

4. RESULTS AND DISCUSSION

The model described above (1)–(9) was compared to simulations of water–steam flow using the following data:

Inlet temperature:	150°C
Outlet pressure:	6.7 bar
Pipe length:	200 m
Pipe diameter:	0.1 m
Mass flow rates:	$6 \leqslant \dot{m} \leqslant 35$ kg s^{-1}
Heat input:	10 kW m^{-1}.

For the analytical model, the fluid properties are taken arbitrarily at the intermediate pressure of 7.5 bar. Figure 2 presents a graph of ΔP against $\dot{m}$ for the mathematical model and for simulation results. It can be seen that the three graphs are qualitatively similar; clearly the possibility of steady-state multiplicity is not an artifact of a particular model.

Figure 3 gives the simulation results using the separated flow model (pressure, temperature and vapor fraction profiles) for three different flow rates with the same overall pressure drop, verifying the steady state multiplicity predicted by the analytical model.

5. CONCLUDING REMARKS

It has been shown that flow of fluid in a heated pipe with a fixed driving head may have more than one steady state

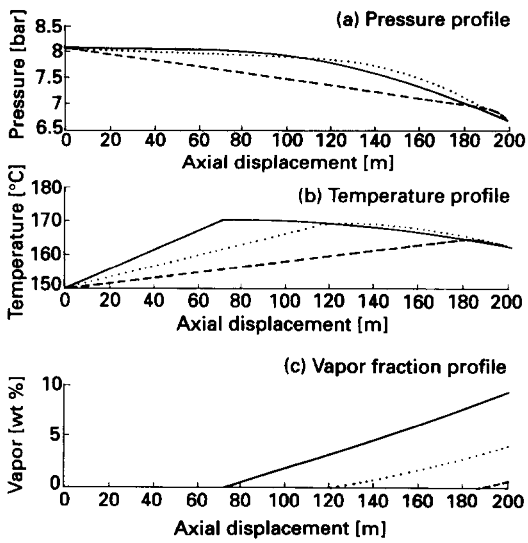

FIG. 3. Simulation results for water–steam flow at three different flow rates: — $\dot{m} = 8.0$ kg s^{-1}, ··· $\dot{m} = 14.0$ kg s^{-1}, --- $\dot{m} = 28.2$ kg s^{-1}. (The separate flow model was used.)

flow rate. The phenomenon was verified by simulations with different models, but is yet to be verified experimentally. As this phenomenon may result in severe operating and control problems, its understanding and verification is important.

Operating problems are expected in cases where the driving head in a system such as presented here is independent of the flow rate as may be the case in oversized centrifugal pumps. In such cases, the extreme sensitivity of the flow rate to pressure changes, and the steady-state multiplicity may result in hysteresis problems and sudden flow rate variations. Particular applications where the observed phenomenon may occur are in natural circulation systems such as thermosyphons and in the cooling tubes of nuclear reactors. Again we should note that a condition for the existence of the phenomenon is that the fluid enters in a subcooled state.

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